Integral Solutions - Integral Two
Anti-Derivative Of The Joint Distribution For A Brownian Motion And Its Maximum And Minimum

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We will define the variable W_t to be the value of a Brownian motion at time t where $W_0 = 0$. Assume that we have a barrier with a value of m that the Brownian motion W_t may or may not cross over the time interval $[0, T]$. Assume also that we have a threshold value w that may be [greater than or equal to] or [less than or equal to] the value of the Brownian motion W_t at time T.

We will define the variable μ to be expected return mean, the variable ϕ to be dividend yield, and the variable σ to expected return volatility. The Brownian motion at time T has the following distribution...

$$W_T \sim N\left[\alpha, v\right] \text{ ...where... } \alpha = \left(\mu - \phi - \frac{1}{2}\sigma^2\right)T \text{ ...and... } v = \sigma^2 T \tag{1}$$

We have the following two barrier problems:

The Maximum Barrier Problem - Probability that W_t reaches or exceeds the maximum barrier m over the time inteval $[0, T]$ and ends up at or below the threshold value w at time T:

$$\text{We want to find... Prob}\left[\text{MAX}\left(W_t\right) \geq m, W_T \leq w\right] \text{ ...where... } m > 0 \text{ ...and... } w \leq m \text{ ...and... } \Delta = +1 \tag{2}$$

The Minimum Barrier Problem - Probability that W_t reaches or falls below the minimum barrier m over the time inteval $[0, T]$ and ends up at or above the threshold value w at time T:

$$\text{We want to find... Prob}\left[\text{MIN}\left(W_t\right) \leq m\right] \text{ ...where... } m < 0 \text{ ...and... } w \geq m \text{ ...and... } \Delta = -1 \tag{3}$$

Problem Setup

We want to solve the following joint density double integral...

$$I = \int\int \frac{2\,(2m - w)}{v\sqrt{2\pi v}} \,\text{Exp}\left\{\frac{\alpha w}{v} - \frac{\alpha^2}{2v} - \frac{1}{2v}(2m - w)^2\right\} \delta m\, \delta w \tag{4}$$

We will define the function $a(m, w)$ as follows...

$$a(m, w) = \frac{2\,(2m - w)}{v\sqrt{2\pi v}} \,\text{Exp}\left\{\frac{\alpha w}{v} - \frac{\alpha^2}{2v} - \frac{1}{2v}(2m - w)^2\right\} \tag{5}$$

Using Equation (5) above we can rewrite Equation (4) above as...

$$I = \int\int a(m, w)\, \delta m\, \delta w \tag{6}$$

To solve the integral above we need to calculate the anti-derivative of the function $a(m, w)$ with respect to the barrier variable m. If we define the function $b(m, w)$ to be the anti-derivative of the function $a(m, w)$ then our goal is the following...

$$\text{We want to define } b(m, w) \text{ ...such that... } a(m, w) = \frac{\delta\, b(m, w)}{\delta m} \tag{7}$$

Finding The Anti-Derivative

Because the anti-derivative of Equation (5) above is calculated with respect to the variable m we will combine all non-m variables into one function. We will define the function $g(w)$ to be the following equation...

$$g(w) = \frac{\alpha w}{v} - \frac{\alpha^2}{2v} \tag{8}$$

Using Equation (8) above we can rewrite Equation (5) above as...

$$a(m,w) = \frac{2\,(2m-w)}{v\sqrt{2\pi v}}\,\text{Exp}\left\{g(w) - \frac{1}{2v}(2m-w)^2\right\} \tag{9}$$

We will define the variable λ to be the following equation...

$$\lambda = 2m - w \;\; \text{...such that...} \;\; \frac{\delta\lambda}{\delta m} = 2 \tag{10}$$

Using Equation (10) above we will define the variable θ to be the following equation...

$$\theta = -\frac{1}{2v}(2m-w)^2 = -\frac{1}{2v}\lambda^2 \;\; \text{...such that...} \;\; \frac{\delta\theta}{\delta\lambda} = -\frac{\lambda}{v} = -\frac{2m-w}{v} \tag{11}$$

Using Equations (10) and (11) above the equation for the derivative of the variable θ with respect to m is...

$$\frac{\delta\theta}{\delta m} = \frac{\delta\theta}{\delta\lambda}\frac{\delta\lambda}{\delta m} = \frac{-2(2m-w)}{v} \tag{12}$$

We will define the function $b(m,w)$ in Equation (7) above as follows...

$$b(m,w) = \frac{1}{\sqrt{2\pi v}}\,\text{Exp}\left\{g(w)\right\}\text{Exp}\left\{\theta\right\} \;\; \text{...where...} \;\; \frac{\delta\, b(m,w)}{\delta\theta} = \frac{1}{\sqrt{2\pi v}}\,\text{Exp}\left\{g(w)\right\}\text{Exp}\left\{\theta\right\} \tag{13}$$

Proof

We will now prove that the function $b(m,w)$ in Equation (13) above is the anti-derivative of the function $a(m,w)$ in Equation (5) above with respect to the barrier variable m. Note that we can rewrite Equation (7) above as.

$$a(m,w) = \frac{\delta\, b(m,w)}{\delta m} = \frac{\delta\, b(m,w)}{\delta\theta}\frac{\delta\theta}{\delta m} \tag{14}$$

Using Equations (11) and (13) above we can rewrite the right side of Equation (14) as...

$$\begin{aligned}\frac{\delta\, b(m,w)}{\delta\theta}\frac{\delta\theta}{\delta m} &= \frac{1}{\sqrt{2\pi v}}\,\text{Exp}\left\{g(w)\right\}\text{Exp}\left\{\theta\right\} \times \frac{-2(2m-w)}{v} \\ &= \frac{-2(2m-w)}{v\sqrt{2\pi v}}\,\text{Exp}\left\{g(w)\right\}\text{Exp}\left\{\theta\right\} \\ &= \frac{2\,(w-2m)}{v\sqrt{2\pi v}}\,\text{Exp}\left\{\frac{\alpha w}{v} - \frac{\alpha^2}{2v} - \frac{1}{2v}(2m-w)^2\right\}\end{aligned} \tag{15}$$

Equation (15) above proves that the function $b(m,w)$ in Equation (13) above is the anti-derivative of the function $a(m,w)$ in Equation (5) above with respect to the barrier variable m. Note that the function $b(m,w)$ can also be written as...

$$b(m,w) = \frac{1}{\sqrt{2\pi v}}\,\text{Exp}\left\{g(w)\right\}\text{Exp}\left\{\theta\right\} = \frac{1}{\sqrt{2\pi v}}\,\text{Exp}\left\{\frac{\alpha w}{v} - \frac{\alpha^2}{2v} - \frac{1}{2v}(2m-w)^2\right\} \tag{16}$$

Appendix

If we take Equation (6) above and multiply that equation by the exponential of the Browian motion variable w then we get the following double integral...

$$I = \int\int \text{Exp}\left\{w\right\} a(m,w)\,\delta m\,\delta w \tag{17}$$

We can rewrite Equation (8) above as...

$$g(w) = w + \frac{\alpha w}{v} - \frac{\alpha^2}{2v} \tag{18}$$

Using Equations (17) and (18) above the anti-derivative of the product of the joint density function $a(m, w)$ and the exponential of w becomes...

$$\begin{aligned} b(m, w) &= \frac{1}{\sqrt{2\pi v}} \operatorname{Exp}\left\{g(w)\right\} \operatorname{Exp}\left\{\theta\right\} \\ &= \operatorname{Exp}\left\{w\right\} \frac{1}{\sqrt{2\pi v}} \operatorname{Exp}\left\{\frac{\alpha w}{v} - \frac{\alpha^2}{2v} - \frac{1}{2v}(2m - w)^2\right\} \\ &= \frac{1}{\sqrt{2\pi v}} \operatorname{Exp}\left\{w + \frac{\alpha w}{v} - \frac{\alpha^2}{2v} - \frac{1}{2v}(2m - w)^2\right\} \end{aligned} \tag{19}$$